THE MOTION OF CONSERVATIVE SYSTEMS IN HARMONIC FORCE FIELDS

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We consider the motion of a conservative system with two degrees of freedom in a plane harmonic force field, i.e. in a field whose potential $V(x, y)$ satisfies the Laplace equation $\Delta V(x, y) = 0$.

The simplest example of such a field is a homogeneous gravitational force field. Another example is the field formed by the logarithmic potential $V = A \ln r$ or the multipole field with the potential $V = Ar^{-n} \cos n\alpha$. It is clear that harmonic fields include those formed by superposition of the above fields. The potential of an arbitrary harmonic field can be expressed in the form $V(x, y) = \operatorname{Re} f(z)$ or $V(x, y) = \operatorname{Im} f(z)$, where $f(z)$ is an arbitrary analytic function of the complex variable $z = x + iy$.

Harmonic force fields occur in various physical problems, e.g. in investigating the motion of charged particles in electric and magnetic fields [1 and 2], in computing of electronic trajectories in electron optics [3 and 4], etc.

1. Let M be the representing point of the system under consideration. If we neglect the relativistic variation of the mass m with changes in velocity v (i.e. if we assume that $v/c \ll \ll 1$, where c is the velocity of light in vacuum), then the problem of finding the trajectories of motion of a point of unit mass ($m = 1$) in an arbitrary conservative field with the potential $V(x, y)$ with an energy constant h reduces to integrating the nonlinear second-order differential equation [5]

$$y'' = (1 + y'^2)\left(-y'\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\right) \qquad \left(\Phi = \ln \sqrt{2(h - V(x, y))}\right) \tag{1.1}$$

Assuming that the potential $V(x, y)$ satisfies the Laplace equation $\Delta V(x, y) = 0$, we infer from the energy integral and from (1.1) that

$$\Delta v^2 = 0, \qquad \Delta\Phi(x, \quad y) = -2(\Phi_x^2 + \Phi_y^2) \tag{1.2}$$

i.e. that the function v^2 is also harmonic, and that $\Phi(x, y)$ belongs to the class of superharmonic functions [6] by virtue of the condition $\Delta\Phi(x, y) < 0$.

We note that the case $\Delta\Phi(x, y) = 0$, $\Delta V < 0$, i.e. the case where $\Phi(x, y)$ is a harmonic function and where the potential $V(x, y)$ belongs to the class of superharmonic functions, yields a hydromechanical analogy [7].

2. Let us reduce differential equation (1.1) of the trajectories to its simplest form. Introducing the angle $\psi(x, y) = \operatorname{arc\,tg} y'$ formed by the velocity vector $\mathbf{v}$ and the positive x-axis, we obtain

$$d\psi(x, y) = \frac{\partial \Phi}{\partial y}dx - \frac{\partial \Phi}{\partial x}dy \qquad (\Phi = \ln v) \tag{2.1}$$

Next, writing

$$d\Phi(x\ \ y)=\frac{\partial\Phi}{\partial x}dx+\frac{\partial\Phi}{\partial y}dy \tag{2.2}$$

and introducing the complex variable $z = x + iy$, we can readily obtain from (2.1) and (2.2) an expression for the logarithm of the complex velocity $\zeta = v \exp(-i\psi)$ of the point M,

$$d\ln\zeta=\frac{\omega(z)\,dz}{v^2}\qquad\left(\omega(z)=-\frac{\partial V}{\partial x}+i\frac{\partial V}{\partial y}\right) \tag{2.3}$$

Here $\omega(z)$ is an analytic function of the complex variable z, this is because one of the Cauchy-Riemann conditions is fulfilled by virtue of the condition of a harmonic force field while the other is fulfilled identically by virtue of the continuity of the partial derivatives.

We denote the integral of the analytic function $\omega(z)$ by $W(z)$, i.e.

$$W(z)=\varphi(x,y)+i\chi(x,y)\qquad\left(W(z)=\int\omega(z)\,dz\right) \tag{2.4}$$

Now, integrating (2.3) and separating the real and imaginary parts, we find by virtue of (2.4) that

$$\varphi(x,\ y)={}^1/_2\,v^2(x,\ y)+\text{const},\qquad d\chi(x,\ y)=-v^2(x,\ y)\,d\psi \tag{2.5}$$

This implies that the lines $\phi(x, y) = \text{const}$ are associated with the equipotential lines $V(x, y) = \text{const}$, and that the lines $\chi(x, y) = \text{const}$ are no longer associated with the isoclines of the trajectory $\psi(x, y) = \text{const}$ as they are in the case of the hydromechanical analogy [7].

3. For an arbitrary harmonic force field with the potential $V(x, y)$ the relationship between the functions $\psi(x, y)$ and $\chi(x, y)$ is given by nonintegrable differential relation (2.5) It is therefore impossible to obtain a closed solution in the general case of an arbitrary harmonic function. The trajectory can be constructed quite accurately by graphic means, however. This involves using the grid formed by the equipotential lines $V(x, y) = \text{const}$ and the lines $\chi(x, y) = \text{const}$ orthogonal to them.

With this method approximation of the trajectory between two neighboring points is linear (in contrast to the Kelvin "radii of curvature" method [8 and 9]) as in [10], the difference being that the lines $\chi(x, y) = \text{const}$ are no longer the isoclines of the trajectories.

Let the initial position of the moving point be A_0 ($z_0 = x_0 + iy_0$), and let its initial velocity be $\mathbf{v}_0 = v_0 \exp(i\psi_0)$.

We construct a ray from the point A_0 at the angle ψ_0 to the x-axis. This ray intersects the line $\chi_1 = \chi(x_1, y_1)$ at the point A_1 ($z_1 = x_1 + iy_1$).

We continue to approximate the trajectory with a broken line by replacing each trajectory arc passing through the two neighboring lines χ_{j+1} and χ_{j+2} by the chord $A_{j+1}A_{j+2}$ at the angle ψ_{j+1} to the x-axis; this angle can be determined by integrating (2.5). Making use of the energy integral and noting that the integrand in the right side of (2.5) is of fixed sign, we apply the mean-value theorem to obtain

$$\psi_{j+1}-\psi_j=\frac{\chi_j-\chi_{j+1}}{2(h-V_j{}^*)}\qquad(j=0,\ 1,\ 2,\ldots) \tag{3.1}$$

where $V_j{}^*$ is the mean value between $V_j = V(x_j, v_j)$ and $V_{j+1} = V(x_{j+1}, y_{j+1})$.

Thus, making use of recursion relation (3.1) and knowing ψ_j (the initial angle of departure ψ_0 is prescribed), we can find ψ_{j+1} and thus construct the points A_{j+1}. Joining these points with a broken line, we obtain the approximate shape of the trajectory.

4. Let us consider the existence of closed trajectories (cycles) in harmonic force fields. To do this we make use of the notion of the quasi-index J_j of the singular point O_j of the force field potential defined as the limiting value of the curvilinear integral taken over a circle (γ) of the small radius r surrounding the singular point O_j [11], i.e.

$$J_j=\frac{1}{2\pi}\lim_{r\to0}\oint_{(\gamma)}\frac{\partial\Phi}{\partial y}\,dx-\frac{\partial\Phi}{\partial x}\,dy \tag{4.1}$$

As we showed in [11], the existence of cycles implies fulfillment of the basic relation

$$1 - J = -\frac{1}{2\pi}\iint_{(\sigma)} \Delta\Phi\, d\sigma \qquad \left(J = \sum_{j=1}^{k} J_j\right) \tag{4.2}$$

where J is the sum of the quasi-indices J_j of the singular points O_j ($j = 1, 2,\ldots, k$) lying inside the closed orbit (C). Integration is carried out over the domain (σ) minus the singular points O_j bounded by the contour (C).

The following theorems are valid for harmonic force fields:

Theorem 1. If some closed trajectory (C) exists in the domain (G) of the force field under consideration, then the sum of the quasi-indices J_j of the singular points O_j lying within (C) in this domain must satisfy the condition

$$-\infty < J < 1 \qquad \left(J = \sum_{j=1}^{k} J_j\right) \tag{4.3}$$

Theorem 2. If the sum of the quasi-indices J_j of the singular points O_j ($j = 1, 2,\ldots, k$) satisfies the condition

$$1 \leqslant J < \infty \qquad \left(J = \sum_{j=1}^{k} J_j\right) \tag{4.4}$$

in the domain (G) of the force field under consideration, then there are no closed trajectories (cycles) in the domain (G).

The proofs of these theorems follow directly from basic relation (4.2) if we recall that the condition $\Delta\Phi < 0$ holds for harmonic force fields.

5. Let us consider some applications. Let the equation of the trajectories in polar coordinates be of the form $r = r(\alpha)$. Making use of the expression

$$\psi = \operatorname{arc\,tg}\frac{r'\operatorname{tg}\alpha + r}{r' - r\operatorname{tg}\alpha} \tag{5.1}$$

familiar to us from differential geometry, we can rewrite differential equation (2.5) of the trajectories in polar coordinates,

$$(r'^2 + r^2)d\chi\,(r,\alpha) = -2\,(h - V\,(r,\alpha))(-rr'' + 2r'^2 + r^2)\,d\alpha \tag{5.2}$$

Here the prime denotes differentiation with respect to α; h is the energy constant.

For example, for a multipole field with the potential

$$V\,(r,\alpha) = -\frac{M}{2\pi}\,\frac{\cos n\alpha}{r^n} \qquad \left(A = \frac{M}{2\pi}\right)$$

where M is the dipole moment, we find from (2.3), (2.4), and (2.5) that

$$\varphi = \frac{A\cos n\alpha}{r^n}, \qquad \chi = -\frac{A\sin n\alpha}{r^n} \qquad \left(W\,(z) = \frac{A}{z^n}\right) \tag{5.3}$$

and differential equation (5.2) of the trajectories on introduction of the new variable $\xi = 1/r$ when the energy constant h is equal to zero becomes

$$n\,(\xi'^2 + \xi^2)\;\;(\xi'\operatorname{tg} n\alpha + \xi) = 2\xi^2\,(\xi'' + \xi) \tag{5.4}$$

Let us investigate the values of the parameter n for which differential equation (5.4) of the trajectories admits of periodic solutions associated with closed orbits (cycles). To do this we compute the quasi-index J of the singular point O ($r = 0$) of the potential $V = -Ar^{-n}\cos n\alpha$. Converting to the polar coordinates $x = r\cos\alpha$, $y = r\sin\alpha$ and noting that $\Phi = \frac{1}{2}\ln\left[2(h + Ar^{-n}\cos n\alpha)\right]$, we obtain

$$\frac{\partial\Phi}{\partial x} = \frac{-An\cos(n+1)\alpha}{2\,(h + A\cos n\alpha\,/\,r^n)\,r^{n+1}}, \qquad \frac{\partial\Phi}{\partial y} = \frac{-An\sin(n+1)\alpha}{2\,(h + A\cos n\alpha\,/\,r^n)\,r^{n+1}} \tag{5.5}$$

On the circle of radius r surrounding the singular point O ($r = 0$) we have $dx = -r\sin\alpha\, d\alpha$, $dy = r\cos\alpha\, d\alpha$, so that by (4.1) and (5.5), on taking the limit as $r \to 0$, we obtain

$$J = \frac{1}{2\pi}\lim_{r\to 0}\int_0^{2\pi}\frac{An\cos n\alpha d\alpha}{2\,(hr^n + A\cos n\alpha)} = \frac{n}{2} \tag{5.6}$$

Hence, for $n \geq 2$ we have $J \geq 1$, which by Theorem 2 ensures the nonexistence of closed orbits (cycles).

Let us consider in more detail the case of a quadrupole ($n = 2$) when the family of equipotential lines $r^2 = C \cos 2\alpha$ (C is a constant) is a family of Bernoulli lemniscates [12] of the four-leaf clover shape shown in Fig. 1.

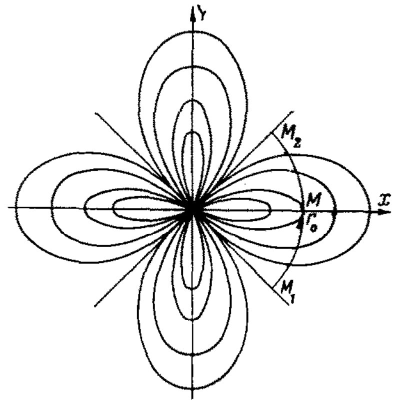

Fig. 1

Differential equation (5.4) of the trajectories becomes

$$\xi' (\xi'^2 + \xi^2) \operatorname{tg} 2\alpha = \xi (\xi\xi'' - \xi'^2) \qquad (5.7)$$

for $n = 2$.

Since the conditions of Theorem 2 ($n = 2$, $J = 1$) are fulfilled here by virtue of (5.6), cycles cannot exist in the force field of a quadrupole, and differential equation (5.7) of the trajectories does not have periodic solutions associated with closed trajectories.

Introducing the new variable $\eta = \operatorname{tg} \mu$, where μ is the angle formed by the tangent to the trajectory and the radius vector $\mathbf{r}$ passing through the point of tangency, noting that $\operatorname{tg} \mu = r/r' = -\xi/\xi'$, integrating (5.7), and clearing logarithms, we obtain the first integral

$$1 + \eta^2 = b \cos 2\alpha \qquad (b = \text{const}) \qquad (5.8)$$

Next, determining $1/\eta = -\xi'/\xi$, integrating, and converting to the variable r, we obtain the equation of the trajectory in polar coordinates,

$$\ln \frac{r}{r_0} = \int_{\alpha_0}^{\alpha} \frac{d\alpha}{\sqrt{b \cos 2\alpha - 1}} \qquad (5.9)$$

Here r_0 and α_0 are the parameters which define the initial position of the representing point of the system. The choice of sign in front of the root is determined by the initial velocity vector $\mathbf{v}_0 = v_0 \exp(i\mu_0)$, since $\eta_0 = \operatorname{tg} \mu_0$.

The integral in the right side of (5.9) can be expressed in terms of Jacobi elliptic functions, introducing a new variable τ

$$\operatorname{tg} \alpha = a \operatorname{cn} \tau, \qquad \operatorname{cn} \tau = \left(\frac{k'}{k}\right)^{1/2} \frac{H_1(\tau)}{\Theta(\tau)} \qquad \left(a^2 = \frac{b-1}{b+1}\right) \qquad (5.10)$$

and using the relationships for Jacobi elliptic functions [13]

$$\operatorname{sn}^2 \tau + \operatorname{cn}^2 \tau = 1, \qquad k^2 \operatorname{sn}^2 \tau + \operatorname{dn}^2 \tau = 1 \qquad \left(k^2 = \frac{a^2}{a^2+1} < 1\right)$$

On integrating we obtain

$$\int \frac{d\alpha}{\sqrt{b \cos 2\alpha - 1}} = -\frac{\tau}{\sqrt{2b}}$$

We thus arrive at the following representation of the equation of the trajectories:

$$\ln \frac{r}{r_0} = \frac{\tau_0 - \tau}{\sqrt{2b}}, \qquad \operatorname{tg} \alpha = \left(\frac{b-1}{b+1}\right)^{1/2} \operatorname{cn} \tau \qquad (5.11)$$

where the initial value of the parameter τ_0 must be determined from the relation $\operatorname{tg} \alpha_0 = a \operatorname{cn} \tau_0$. If we set $\alpha_0 = 0$ (this can always be done by rotating the axes), it turns out that $\operatorname{cn} \tau_0 = 0$.

Hence, $\tau_0 = K$, where $K(k)$ is the value of the total elliptic integral of the first kind with the modulus $k = \sqrt{a^2/(1 + a^2)}$.

We note that $\xi = 1/r = \text{const}$ is also a solution of (5.7). This result can be obtained by considering the first integral of (5.8) which we represent in the form $\cos^2 \mu \cos 2\alpha = \cos^2 \mu_0$ and assume that the angle of departure is $\mu_0 = \pi/2$.

Analysis based on consideration of the equations of motion in the field of a quadrupole,

$$r^{\cdot\cdot} - r\alpha^{\cdot 2} + \frac{2A\cos 2\alpha}{r^3} = 0\,, \qquad \frac{d}{dt}(r^2\alpha^{\cdot}) + \frac{2A\sin 2\alpha}{r^3} = 0$$

indicates that under the initial conditions $t = 0$, $\alpha_0 = 0$, $\mu_0 = \pi/2$, $h = 0$, $r = r_0$ the solution $r = \text{const}$ corresponds to the oscillatory motion of the representing point M along the arc M_1M_2 of a circle of radius $r = r_0$ with the angular amplitude $\alpha = \pm/4$ (see Fig. 1).

The points M_1 and M_2 at which the velocity of the representing point M becomes zero are cusps.

The angular velocity ω, the angular acceleration ε, and the oscillation period T are given by

$$T = \frac{4}{\sqrt{B}} \int_0^{\pi/4} \frac{d\alpha}{\sqrt{\cos 2\alpha}} \qquad \left(\omega^2 = B\cos 2\alpha, \quad \varepsilon = -B\sin 2\alpha, \quad B = \frac{2A}{r_0^4}\right)$$

By making the substitution $\operatorname{tg}\alpha = \operatorname{cn}\tau$ we can express the above integral in terms of Jacobi elliptic functions. Omitting the computations and making use of the value for B, we finally obtain the following expression for the period

$$T = \frac{2K}{\sqrt{A}}\, r_0^2 \qquad \left(A = \frac{M}{2\pi}\right) \tag{5.12}$$

Here M is the dipole moment and $K(k)$ is the value of the total elliptic integral of the first kind with the modulus $k = \sqrt{1/2}$. It is clear from this that the period T depends on the dipole moment M and on the initial radius of departure r_0.

BIBLIOGRAPHY

1. Buchholz, H., Elektrische und magnetische Potentialfelder. Springer, Berlin, 1957.
2. Kapitsa, P.L., High-Power Electronics. Izd. Akad. Nauk SSSR, Moscow, 1962.
3. Glaser, W., Grundlagen der Elektronenoptik. Springer, Wiesbaden, 1952.
4. Cosslett, V.E., Introduction to Electron Optics. Oxford University Press, 1946.
5. Belen'kii, I.M., Introduction to Analytical Mechanics. Izd. "Vysshaya shkola", Moscow, 1964.
6. Privalov, I.I., Subharmonic Functions. ONTI SSSR, Moscow-Leningrad, 1937.
7. Belen'kii, I.M., A new analogy in mechanics. Dokl. Akad. Nauk SSSR Vol. 140, No. 6, 1961.
8. Kelvin, On graphic solution of dynamical problems. Phil. Mag. Ser. 5, Vol. 32, No. 210, 1892.
9. Sansone, G., Ordinary Differential Equations, Vol. 2, Izd. inostr. lit., Moscow, 1954.
10. Belen'kii, I.M., A graphic method of constructing trajectories. Izv. Akad. Nauk SSSR OTN, Mekhanika i mashinostroenie No. 5, 1962.
11. Belen'kii, I.M., On sufficient conditions for the absence of periodic trajectories in conservative systems. PMM Vol. 30, No. 3, 1966.
12. Savelov, A.A., Plane Curves. Fizmatgiz, Moscow, 1960.
13. Whittaker, E.T. and Watson, G.N., A Course in Modern Analysis, Vol. 2, Gostekhizdat, 1934.

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